PERIODICA POLYTECHNICA SER. MECH. ENG. VOL. 47, NO. 1, PP. 81–89 (2003)

FILLING TRIANGULAR HOLES BY CONVEX COMBINATION OF SURFACES

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Received: October 24, 2002

Abstract

A surface generation method is presented based on convex combination of surfaces with rational weight functions. The three constituents and the resulting surface are defined over the same triangular domain. The constructed surface matches each component along one of its boundary curves with C^0 or C^1 continuity depending on the weight functions in the combination. The method can be applied in surface modelling for filling triangular holes.

Keywords: filling holes, triangular surface, convex combination.

1. Introduction

Filling gaps and holes is a crucial problem in surface modelling, and a number of algorithms are known and implemented based on different concepts.

A classical method is to generate Coons-type surfaces from boundary curves and derivative values along these curves in order to get a satisfying smooth surface patch which matches all these boundary data [3]. An essential extension of Coons' construction is given in [11, 12] using also surface patches as input data instead of 'wire-frame' data. The constituents and the resulting surface are defined over a rectangular domain and the triangular patches are generated as degenerate rectangular ones. Trigonometric [9, 10], rational [12] and Hermite [11] weight functions (also called blending functions) have been used and C^2 continuous connection to the surrounding surfaces of the hole has been achieved. A Coons-type scheme of interpolation is described in [2] from boundary and derivative values of a function defined on a triangle with rational, trilinear and tricubic blending functions. The case of first degree rational blending functions can be found also in textbooks [4].

A different concept is frequently used for interpolation to scattered data. First, the domain of the given points is tesselated into triangles or tetrahedra, then a smooth function of two or more variables is constructed which assumes given values and given cross boundary or normal derivatives on the boundary of each triangle or tetrahedron. A piecewise rational scheme is described in [1] that interpolates to function and gradient values at vertices of tetrahedra, and is continuously differentiable everywhere in the underlying domain. First, underlying cubic Hermite interpolants are constructed on all faces and a convex combination is built with quadratic rational weight functions.

The further mentioned interpolation methods are developed for a triangulated set of points. Let $V_1V_2V_3$ denote a planar triangle, and $F(x_i, y_i)$ $(i = 1, 2, 3)$ be function values at the vertices. In order to get smooth resulting surface, given or estimated values of the partial derivatives $F_x(x_i, y_i)$ and $F_y(x_i, y_i)$ at the vertices are also used in the interpolations. Let $D_i[F]$ be an 'underlying interpolant' which matches position and directional derivative values of F at the vertex V_i and at the point S_i, $(i = 1, 2, 3)$ on the opposite side of the triangle (*Fig.1*). Methods working with such functions are called 'side-vertex' methods in the literature.

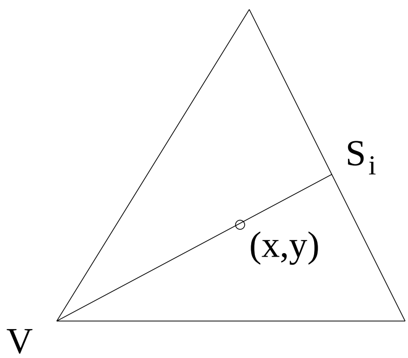

Fig. 1. Side-vertex interpolation

A commonly used combination of the constructed interpolants D_i is

$$\sum_{i=1}^{3} W_i(x, y) D_i[F](x, y)$$

with the weight functions

$$W_i = \frac{b_j^2 b_k^2}{b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2}, \quad i \neq j \neq k, \tag{1}$$

where $b_1,\ b_2\,, b_3$ are the barycentric coordinates of the point (x, y) [8].

There are several methods for estimating the partial derivatives from the scattered data $F(x_i, y_i)$ and for selecting the cross boundary derivatives on the triangle edges which are needed in surface generation. In [5] and [6] such methods are proposed, and lower degree weight functions

$$W_i(x, y) = \frac{b_j b_k}{b_1 b_2 + b_1 b_3 + b_2 b_3}\,, \quad i = 1, 2, 3\,, \quad i \neq j \neq k \tag{2}$$

are used in the combination of the underlying interpolants.

In [7] linear and cubic Hermite side-vertex interpolants are constructed, and the weight functions

$$W_i = \frac{1}{b_i^2} \Big/ \left(\frac{1}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_3^2} \right), \tag{3}$$

or the lower degree counterparts

$$W_i = \frac{1}{b_i} \Big/ \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} \right) \tag{4}$$

are used.

In this paper instead of Coons' method we shall follow the second ideas based on convex combination of surfaces.

2. Surface Construction

The filling of a three-sided hole with a surface joining continuously to the surrounding surfaces is known as the suitcase corner problem. A number of solutions have been published already using subdivision algorithms from generated data along the boundary of the hole or constructing surface patches represented by spline functions of different types.

The presented surface construction uses convex combination of three surfaces which are either extensions of the surfaces surrounding the hole, or are constructed in such a way that each joins smoothly to one of the three given surfaces along one boundary curve of the hole. Such constructions are described in textbooks, and they are not the subject of this paper.

Combining surfaces instead of 'wire-frame data' yields that not only the boundary data, but also the shape of these components influence the resulting surface. To the best of the author's knowledge applying such surface construction for filling holes is novel in CAD literature.

Let us assume that the differentiable surfaces $\mathbf{r}_i(u, v)$, $(i = 1, 2, 3)$ are given over the 'standard triangle' Δ with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ in the uv plane. The three boundary curves $\mathbf{r}_1(u, 0)$, $\mathbf{r}_2(0, v)$ and $\mathbf{r}_3(u, 1 - u) = \mathbf{r}_3(1 - v, v)$ $(0 \le u \le 1,\ 0 \le v \le 1)$ have common endpoints, i.e. they form the boundary of a curvilinear spatial triangle. The barycentric coordinates of the point (u, v) are $b_1 = v$, $b_2 = u$, $b_3 = 1 - u - v$ (*Fig. 2*).

The investigation of weight functions in convex combinations constructed for interpolating to scattered data has shown that those in formulae (1), (2), (3) and (4) are suitable for our purpose. The combination of three constituents with these blending functions matches the three boundary curves of the curvilinear triangle. Applying the higher order blending functions in (1) and (3), it matches also the cross boundary derivatives of the corresponding component along the common boundary line.

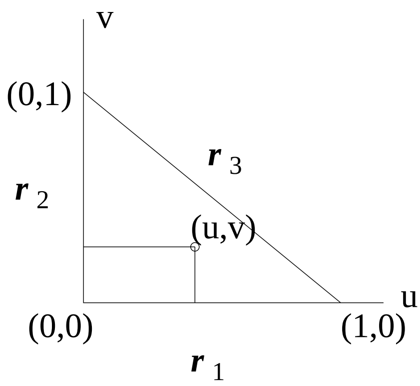

Fig. 2. Triangular parameter domain

Theorem 1 *The surface composed from the differentiable surfaces* $\mathbf{r}_i(u, v)$, $(i = 1, 2, 3)$, $(u, v) \in \triangle$ *satisfying*

$$\mathbf{r}_1(0, 0) = \mathbf{r}_2(0, 0), \quad \mathbf{r}_2(0, 1) = \mathbf{r}_3(0, 1), \quad \mathbf{r}_3(1, 0) = \mathbf{r}_1(1, 0)$$

by the vector function

$$\mathbf{r}(u, v) = \sum_{i=1}^{3} W_i(u, v)\mathbf{r}_i(u, v), \quad (u, v) \in \triangle, \tag{5}$$

matches the three boundary curves, i.e. $\mathbf{r}(u, 0) = \mathbf{r}_1(u, 0)$, $\mathbf{r}(0, v) = \mathbf{r}_2(0, v)$ *and* $\mathbf{r}(u, 1-u) = \mathbf{r}_3(u, 1-u)$, $u \in [0, 1]$, $v \in [0, 1]$. $W_i(u, v)$ *are weight functions in (1), (2), (3) or (4), and* $b_1 = v$, $b_2 = u$, $b_3 = 1 - u - v$ *are the barycentric coordinates of the point* (u, v).

Moreover, in the cases of the weight functions in (1) and in (3) the corresponding cross boundary derivatives of $\mathbf{r}(u, v)$ *and the constituents are also equal along these common boundary lines.*

The proof of these statements can be obtained by simple computations and derivations.

3. Examples

The three surfaces to be combined are generated as quadratic Bézier surfaces over the triangular parameter domain shown in *Fig. 2*. The surface $\mathbf{r}_1(u, v)$ is a curved triangle with one side in the xy plane and a vertex on the axis z. The surface $\mathbf{r}_2(u, v)$ is a planar patch in the coordinate plane xz with one vertex in the origin. The third one, $\mathbf{r}_3(u, v)$ has a boundary curve in the coordinate plane yz and a vertex in front (*Fig. 3*).

The first combination has been computed with weight functions in (1), and is shown in two different projections in *Figs.* 4 and 5.

In *Fig.* 6 two extended constituents are also shown together with the generated surface. The third component is lying in front of the result, therefore it is not drawn. The tangential connection can be seen clearly.

The use of lower degree weight functions in the combination yields different shape and only C^0 connection to the components along the boundary curves (*Fig.* 7).

In *Fig.* 8 a surface is shown constructed only from the boundary curves of the three input surfaces as a convex combination with weight functions in (1). Comparing the shape with that in *Fig.* 7, the influence of surface components can be seen clearly. However, the surface in *Fig.* 7 does not match the derivatives of the components.

For illustrating the effect of a different surface component with unchanged boundary data, we replaced the planar surface patch with a narrower one. The resulting surface is shown in *Fig.* 9.

Finally, the same surfaces are combined with the weight functions in (3) which yield C^1 connection along the boundary curves (*Fig.* 10).

The computations and the *Figs.* 3–10 have been made by the algebraic program package Maple.

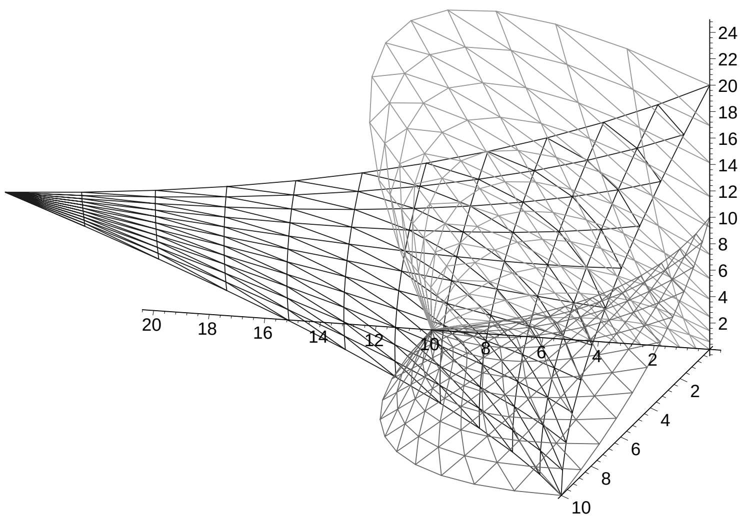

Fig. 3. The three input surfaces

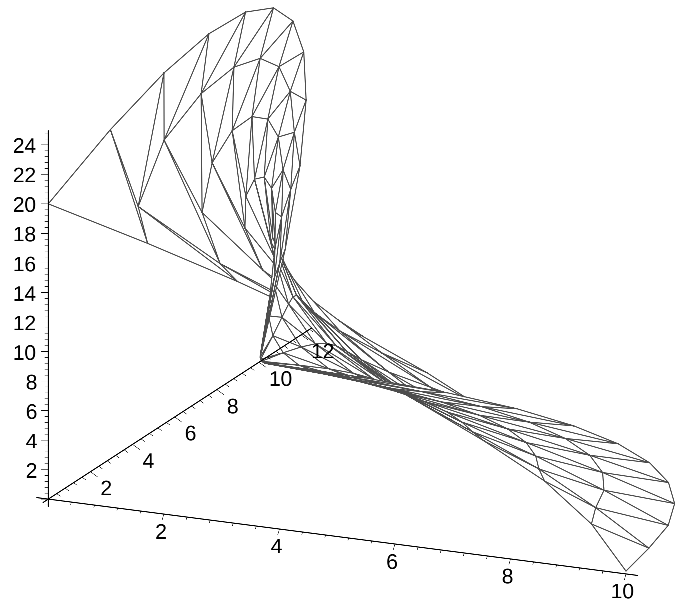

Fig. 4. The resulting surface with blending functions in (1)

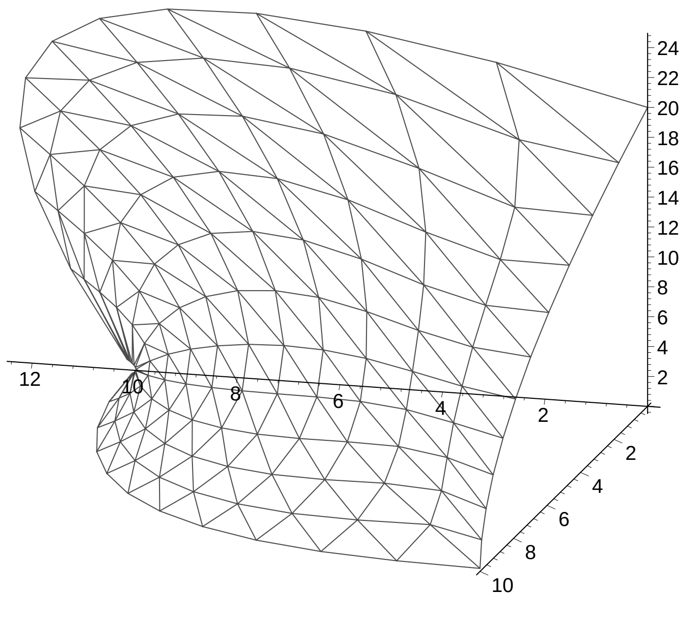

Fig. 5. The same result from a different view

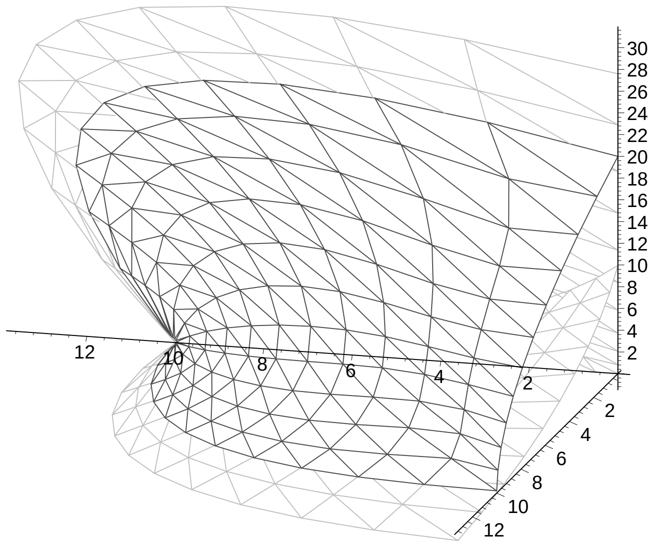

Fig. 6. The resulting surface and two extended constituents

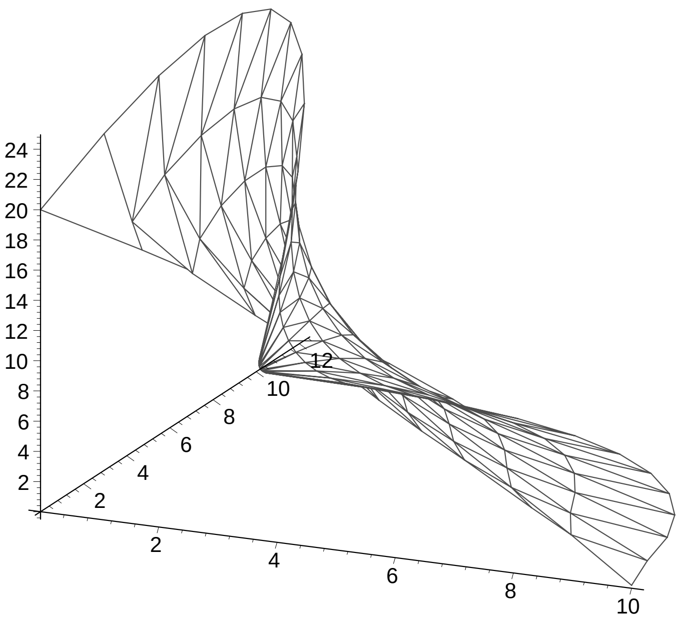

Fig. 7. The combination with lower degree weight functions

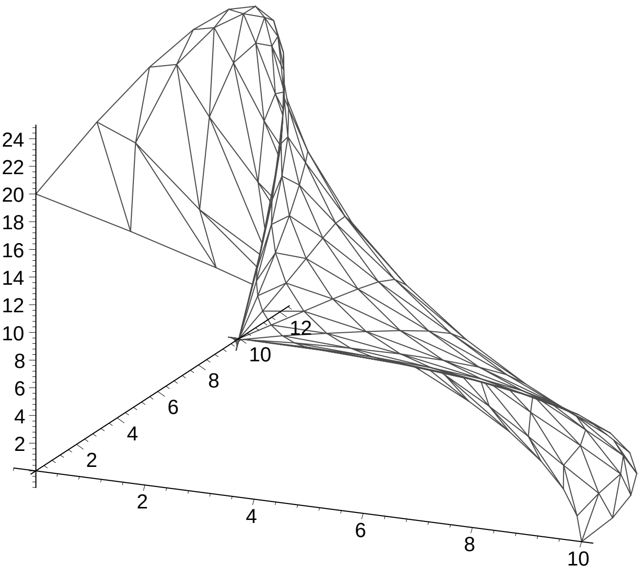

Fig. 8. Combination of the three boundary curves

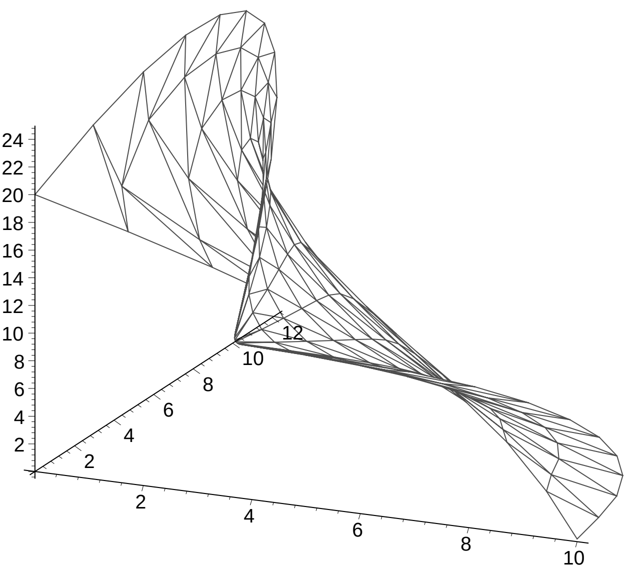

Fig. 9. Unchanged boundaries with one changed constituent

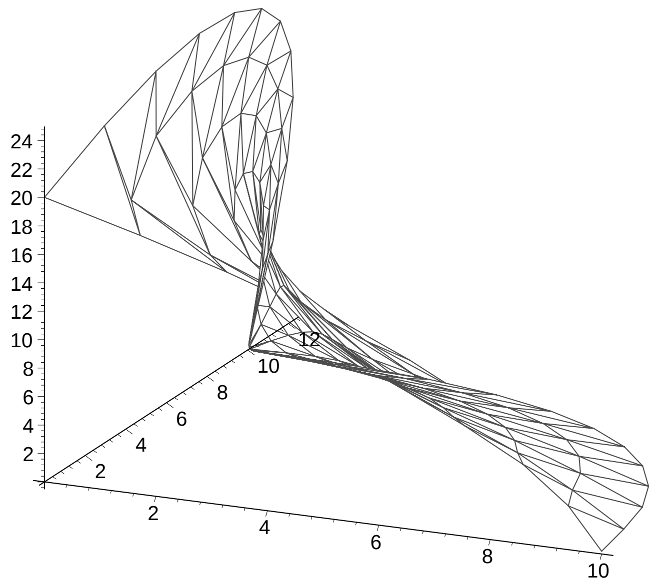

Fig. 10. The surfaces in *Fig. 3* blended with functions in (3)

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